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The Pattern of Eigenfrequencies of Overtones of Torsional Oscillations of a Layered Spherical Shell

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In torsional oscillations of a layered spherical shell, the eigenfrequencies ${}_m\omega_n$ of overtones, for given angular order n, lie near to $m\pi/\gamma$, where m is the number of the overtone and γ the time taken for a shear wave to travel through the shell along a radius. The graph of $\gamma{}_m\omega_n/\pi - m$ against m is here called the *pattern of eigenfrequencies*. It is shown in a numerical experiment that this pattern is very sensitive to variations in layer thickness.

1. INTRODUCTION

In 1959 Alterman, Jarosh, and Pekeris [1] published a very useful analysis of the problem of free oscillations of a sphere. For purely torsional axially-symmetric oscillations of angular frequency ω_n of a spherically symmetric, nonrotating, perfectly elastic isotropic sphere of density $\rho(r)$ and rigidity $\mu(r)$, where r is radius, the spherical polar coordinates (r, θ, ϕ) separate, and the displacement at a general point may be expressed as

$$(0, 0, -\partial\psi/\partial\theta), \tag{1.1}$$

where

$$\psi = W(r)\, P_n(\cos\theta), \tag{1.2}$$

P_n being a Legendre polynomial, and $W(r)$ a solution of

$$\mu\left(W'' + \frac{2}{r}W'\right) + \mu'\left(W' - \frac{1}{r}W\right) + \left\{\rho\omega_n{}^2 - \frac{n(n+1)}{r^2}\mu\right\} W = 0. \tag{1.3}$$

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The only nonzero component of stress is

$$\widehat{r\phi} = \widehat{\phi r} = \mu \left(W' - \frac{1}{r} W \right) \frac{\partial}{\partial \theta} P_n(\cos \theta). \tag{1.4}$$

If we consider a vibrating shell, free from stress at its bounding surfaces $r = a$, $r = b$, the eigenfrequencies of overtones corresponding to a specified (Legendre) order n are those of the *Sturm–Liouville system:*

$$\mu \left(W'' + \frac{2}{r} W' \right) + \mu' \left(W' - \frac{1}{r} W \right) + \left\{ {}_m\omega_n{}^2 \rho - \frac{n(n+1)}{r^2} \mu \right\} W = 0 \tag{1.5}$$

in $a < r < b$, with

$$W' - \frac{1}{r} W = 0 \qquad \text{at} \quad r = a, \quad r = b. \tag{1.6}$$

This system can be reduced to normal form by the *Liouville transformation:*

$$W = rZ/M, \qquad t = \int_a^r dr/\beta(r), \tag{1.7}$$

$$\beta = \mu^{1/2}\rho^{-1/2}, \qquad M = r^2\rho^{1/2}\beta^{1/2}. \tag{1.8}$$

Then, provided μ does not vanish in (a, b) and μ, μ' are continuous in (a, b), it can be shown that the angular frequencies ${}_m\omega_n$ of overtones are asymptotically given, for fixed n and large m, by

$${}_m\omega_n{}^2 = \left(\frac{m\pi}{\gamma} \right)^2 + \frac{A}{\gamma^2} + \frac{B}{\gamma^2 m^2} + O\left(\frac{1}{m^3} \right), \tag{1.9}$$

where $\gamma = \int_a^b dr/\beta$ and A and B are independent of m (see Anderssen, Cleary, and Osborne [2]).

This statement is inadequate in two ways. On the one hand, Anderssen and Cleary showed in 1974 [3] that angular frequencies computed for certain Earth-models do not fit well into formula (1.9), and that the fit becomes worse if discontinuities of μ and ρ are accentuated. On the other hand, the restrictions needed on μ and ρ for (1.9) to hold [6] are too severe for seismologists, who want to consider Earth-models with internal surfaces of discontinuity.

McNabb, Anderssen, and Lapwood [8] showed that if the Sturm–Liouville coefficients are discontinuous the formula (1.9) no longer applies; then ${}_m\omega_n/(m\pi/\gamma)$, instead of approaching asymptotically to unity, displays an irregular pattern indefinitely repeated. When there is only one discontinuity, the graph of ${}_m\omega_n/(m\pi/\gamma)$ is a sinusoidal curve. For this reason the effect was called a "solotone effect."

Sato and Lapwood have examined the solotone effect for spherical shells made up of uniform layers, for which the frequency equation can be obtained exactly in terms of spherical Bessel Functions [9, 10, 7]. They have shown that the *amplitude* of the oscillations in ${}_m\omega_n/(m\pi/\gamma)$ is determined by the magnitude of the discontinuities, but the *period* and *phase* by the thicknesses and wave-speeds of the layers. In this they confirm work of Anderssen [4] and Wang, Gettrust, and Cleary [11]. These authors have also shown that the solotone effect depends for its existence on internal reflections and resonances. It appears that for a given layering the pattern of eigenfrequencies is very sensitive to small changes in the thicknesses of the layers.

In this paper we examine, with the help of a numerical experiment, the effect of small changes—in layer thicknesses alone—from a standard model. The standard is an averaged PEM-A, described by Dziewonski, Hales, and Lapwood [5]. We confirm that remarkable changes in the "eigenfrequency pattern" accompany very moderate changes in relative thicknesses of the layers of a three-layer shell.

2. Formula for the Solotone Deviation S

Consider a three-layer shell composed of three uniform spherical layers, as follows:

$$\begin{array}{llll} \text{(3) top layer} & r_3 > r > r_2\text{, density } \rho_3\text{,} & \text{shear velocity } \beta_3, & \\ \text{(2) middle layer} & r_2 > r > r_1 \quad \rho_2 & \beta_2, & \\ \text{(3) bottom layer} & r_1 > r > r_0 \quad \rho_1 & \beta_1. & \end{array} \tag{2.1}$$

If R_1 is the reflection coefficient for a ray from region (1) impinging normally on the (1, 2) boundary, then

$$R_1 = (\rho_1\beta_1 - \rho_2\beta_2)/(\rho_1\beta_1 + \rho_2\beta_2).$$

Similarly we define R_2 and obtain

$$R_2 = (\rho_2\beta_2 - \rho_3\beta_3)/(\rho_2\beta_2 + \rho_3\beta_3). \tag{2.2}$$

If we define

$$\chi_p = \omega \int_{r_{p-1}}^{r_p} dr/\beta, \qquad p = 1, 2, 3, \tag{2.3}$$

and R_1', R_2' as the reflection coefficients for *oblique* impact, then the frequency equation for overtones with the same angular order has been shown to be [10]:

$$\sin(\chi_1 + \chi_2 + \chi_3) = R_1' \sin(\chi_1 - \chi_2 - \chi_3) + R_2' \sin(\chi_1 + \chi_2 - \chi_3) - R_1'R_2' \sin(\chi_1 - \chi_2 + \chi_3). \tag{2.4}$$

A zero-order approximation to (2.4) is obtained by neglecting R_1', R_2'; it is

$$\chi_1 + \chi_2 + \chi_3 = m\pi, \qquad m = 1, 2, 3,\dots . \tag{2.5}$$

Then a first-order approximation comes from

$$\chi_1 + \chi_2 + \chi_3 = m\pi + \delta, \tag{2.6}$$

where δ is small, and the use of R_1, R_2 as approximations for R_1', R_2'. Neglecting terms of order R_1R_2 we get [7]

$$S \equiv \gamma_m\omega_n/\pi - m \doteqdot (R_1/\pi)\sin 2\chi_1 - (R_2/\pi)\sin 2\chi_3 \equiv S', \tag{2.7}$$

where

$$\gamma = (\chi_1 + \chi_2 + \chi_3)/{}_m\omega_n\,. \tag{2.8}$$

The expression S', which represents S approximately as the superposition of two sinusoidal terms, gives a clearer view of its structure than do computed values of S, which may be obtained from precisely computed eigenfrequencies. It turns out that the values of S and S' are rather insensitive to n, the (Legendre) order of the mode of oscillation. This is because when m is large the wavelength is $2\pi\beta/{}_m\omega_n \doteqdot (a - b)/m$, which is much smaller than the characteristic wavelength of the lateral variation of the mode, which is approximately b/n. Thus the wave does not feel the sphericity of the boundary constraint, behaving like a plane wave.

We therefore compute S after neglecting Earth-curvature, and for the investigation of pattern changes we use the approximations

$$\chi_p \doteqdot \frac{r_p - r_{p-1}}{\beta_p}\frac{m\pi}{\gamma}, \qquad p = 1, 2, 3. \tag{2.9}$$

We adopt as our standard model A (which is to be perturbed) the averaged PEM-A, which has the following parameters [5, 10]:

r_3 : 6368 km,	β_3 : 4.51 km/s,	ρ_3 : 3.42 gm/cc,	$R_2 = 0.1516$,
r_2 : 5951 km,	β_2 : 5.30 km/s,	ρ_2 : 3.95 gm/cc,	$R_1 = 0.2367$,
r_1 : 5701 km,	β_1 : 6.77 km/s,	ρ_1 : 5.01 gm/cc,	
r_0 : 3485.7 km.			

Figure 2 shows the distribution of values of S for model A for $m = 1, 2, 3,..., 60$. The graph of S', which we have not included here, differs very little from Fig. 2, except for $m < 6$.

We observe two particular features of Fig. 2:

(a) there is a strong periodicity, of period 10 in m, the repetition of pattern with each increase of 10 in m being almost exact (the periodicity shows in bold zig-zag lines);

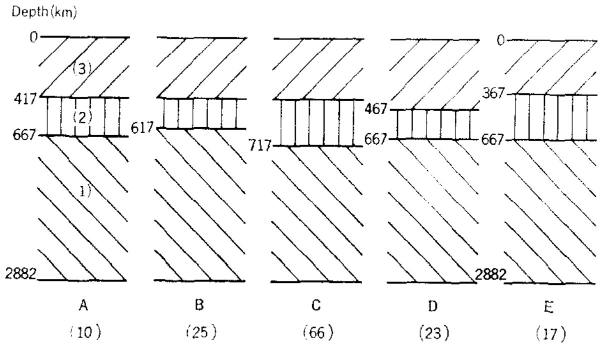

FIG. 1. Scheme of layer thicknesses in Models A, B, C, D, E; A being an averaged PEM-A. Depths of discontinuities are shown on the left side of each section. Figures in brackets below the letters A to E show the recurrence periods (in m) of the patterns of eigenfrequencies for the five models.

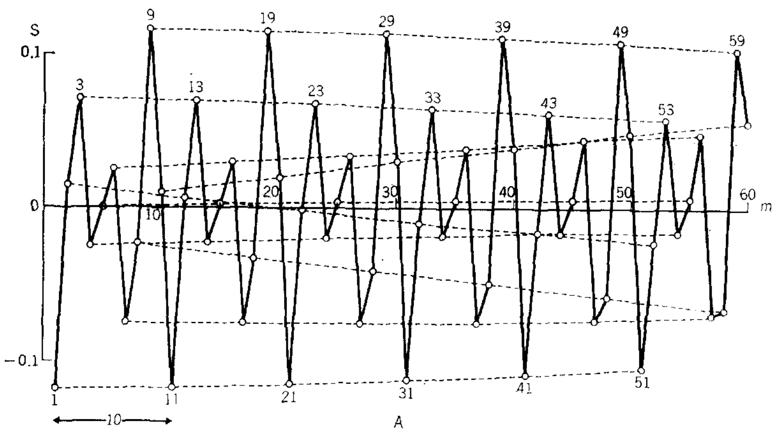

FIG. 2. Graph of S against m for Model A; the recurrence period of 10 in m shows clearly. The ten almost flat lineations in the pattern of eigenfrequencies are shown by dashed lines. That beginning at $m = 1$ carries the numbers ($m = 1, 11, 21, 31, 41, 51$) of the modes whose frequencies lie on it. Similarly for the lines starting at $m = 3$ and $m = 9$ (to avoid confusion the other dashed lines carry no numbers).

(b) alternatively each point lies on one of ten flat curves (dashed lines). Each such curve is close to a straight line nearly parallel to the axis of m. If the point corresponding to $m = m_1$ lies on a particular curve, then so do points corresponding to $m = m_1 + 10, m_1 + 20, m_1 + 30, \ldots$.

This combination of almost periodic (zig–zag) pattern and alignments we will call the *pattern of eigenfrequencies.*

3. Changes of Pattern of Eigenfrequencies due to Small Changes in Layer Thickness

We now make slight shifts in the surfaces of discontinuity, leaving all other parameters and overall thickness unchanged. The model defined in Section 2 is *A*. In *B* the lower discontinuity is raised by 50 km, in *C* it is lowered by 50 km. In *D* and *E* the lower discontinuity is not shifted, but in *D* the upper discontinuity is lowered by 50 km, in *E* it is raised by 50 km. The five arrangements of layers are shown schematically in Fig. 1.

For each model the values of S corresponding to $m = 1, 2, 3, \ldots, 60$ have been calculated from (2.7); the results are exhibited in Figs. 2–6, which enable us to compare

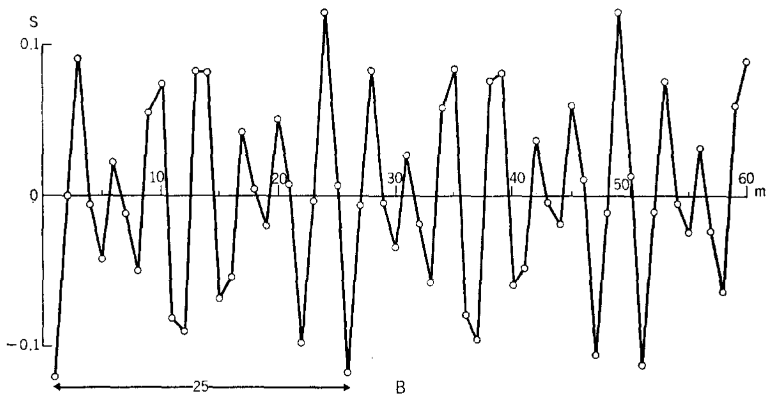

Fig. 3. Graph of S against m for Model B; the recurrence period is 25.

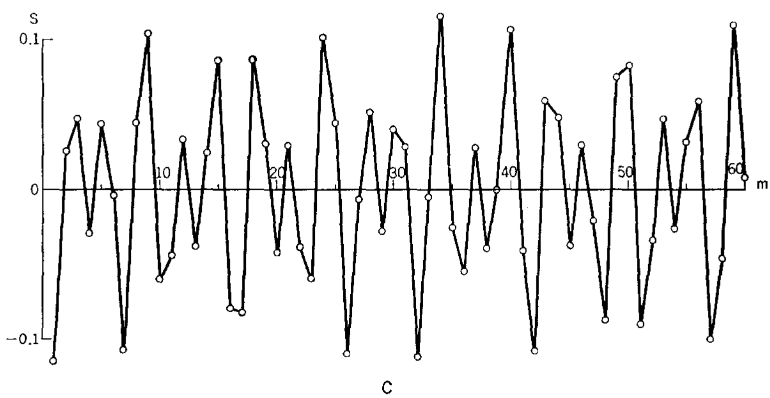

Fig. 4. Graph of S against m for Model C. After an interval of 33 in m the pattern of eigenfrequencies repeats with reversed sign: this can be seen in the graph. The recurrence period of 66 is too long to show in the graph.

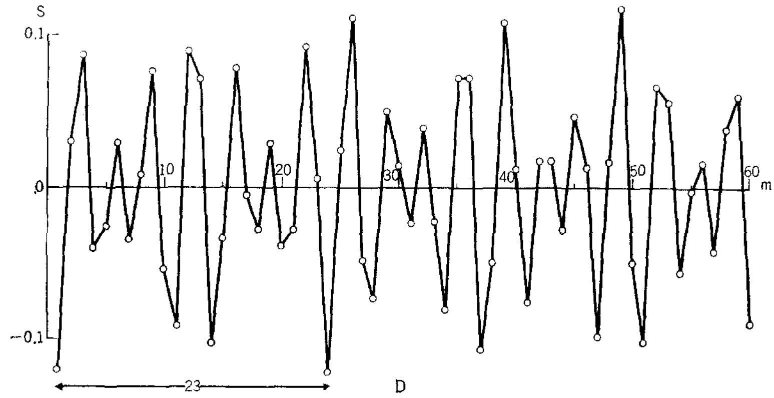

FIG. 5. Graph of S against m for Model D; the recurrence period is 23.

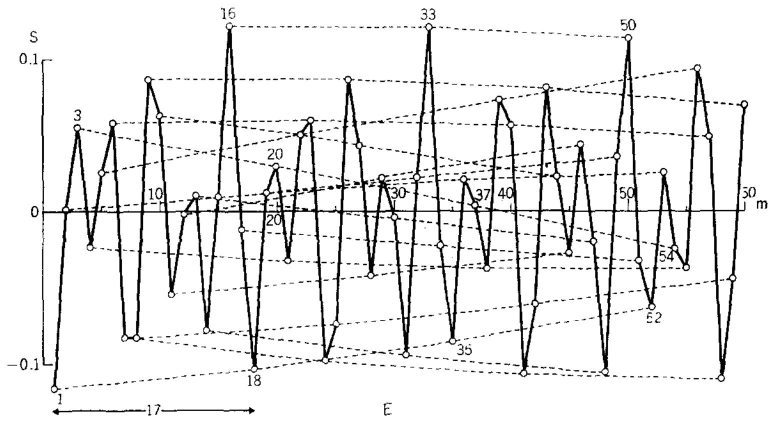

FIG. 6. Graph of S against m for Model E; the recurrence period is 17. The 17 dashed lines mark lineations: they are not as flat as those in Fig. 2 (Model A), for reasons given in the text. The dashed line starting at $m = 1$ carries the numbers of those models whose frequencies lie on it ($m = 1$, 18, 35, 52). Similarly for the lineations starting at $m = 3$ and $m = 16$.

the patterns of eigenfrequencies for A, B, C, D and E. First we notice the remarkable changes in recurrence period in m. This period is 10 for A, 25 for B, 23 for D, and 17 for E. At first inspection there seems to be no recurrence in C; but closer observation shows that values do repeat—though with opposite sign—when m is increased by 33. Thus a recurrence period of 66 exists, and would be seen if a longer series of values of m were used.

We can check these observed values of recurrence period in m. Lapwood and Sato have pointed out [7] that if we write $2\chi_1 = m\alpha_1$, $2\chi_3 = m\alpha_2$, to define α_1 and α_2, then there will be recurrence with period k in m if there is an integral k such that both $k\alpha_1$ and $k\alpha_2$ are nearly integral multiples of 2π. The most satisfactory procedure is to list $k\alpha_1/\pi$ and $k\alpha_2/\pi$ for $k = 1, 2, 3,...$ and pick out an appropriate k. But in order to save space we quote only the values of k now under discussion for the five models. We find the following results; it is clear that the values given for $k\alpha_1/\pi$ and $k\alpha_2/\pi$ deviate only slightly from even integers.

Model	k	$k\alpha_1/\pi$	$k\alpha_2/\pi$
A	10	14.018	3.961
B	25	35.994	9.946
C	66	90.037	26.029
D	23	32.128	10.167
E	17	23.916	5.947

The alignments corresponding to these values of k are shown in Figs. 2 and 6, but not in the other figures, since the addition of large numbers of roughly parallel line segments would make the figures confusing. We may note that the fact that $k\alpha_1/\pi$ and $k\alpha_2/\pi$ diverge more from even integers for E than for A means that the alignments in E diverge further from parallels to the m-axis than for A.

4. Conclusions

We have shown that, in this problem of the distribution of eigenfrequencies of overtones of torsional oscillations of a layered spherical shell, the *pattern of eigenfrequencies* is remarkably sensitive to small changes in the thickness of the layers. This happens because the pattern of eigenfrequencies is controlled by resonances between pulses reflected between surfaces of discontinuity.

Thus if observational data were sufficiently refined the pattern of eigenfrequencies might become an excellent discriminator between proposed Earth-models with different layering, in so far as the models assumed first-order discontinuities at inner boundaries.

The phenomenon which we have investigated here is common to all physical phenomena for which a sequence of eigenvalues arises from a Sturm–Liouville system with discontinuous coefficients. The spheroidal oscillations of a nonuniform spherical shell provide another example.

We have much satisfaction in dedicating this paper to the memory of Professor Zipora Alterman, for it was in such exercises combining mathematical analysis and computing that she excelled and delighted.

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